POSITIVE SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS WITH CONCAVE AND CONVEX TERMS

by

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a C^2-boundary $\partial\Omega$.

We consider the following nonlinear Neumann problem:

$$\left\{\begin{array}{ll} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{q-2}u(z) + f(z, u(z)) & \\ \text{a.e. in } \Omega, \quad u > 0, & (1) \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, & \\ \beta \in L^\infty(\Omega)_+ \setminus \{0\}, \quad \lambda > 0, \quad 1 < q < p < \infty. & \end{array}\right.$$

Here $\Delta_p u = \text{div}\,(||Du||^{p-2}Du)$.

Note that the term $x \to \lambda|x|^{q-2}x$ is $(p-1)$-*sublinear* near $+\infty$, i.e.

$$\lim_{x\to+\infty} \frac{\lambda x^{q-1}}{x^{p-1}} = 0$$

("concave" term).

The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ is supposed to be $(p-1)$-*superlinear* near $+\infty$ in x, i.e.

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$$

("convex" perturbation).

The aim of this work is to establish a *bifurcation - type* result for the positive smooth solutions of (1), with respect to the parameter $\lambda > 0$.

Particular case: The right hand side term of (1) has the form $x \to \lambda|x|^{q-2}x + |x|^{r-2}x$, with

$$1 < q < p < r < p^* = \begin{cases} \dfrac{Np}{N-p}, & \text{if } p < N \\ +\infty, & \text{if } p \geq N \end{cases}.$$

This particular case is what we mostly encounter in the literature and only in the context of Dirichlet problems.

In this direction we mention the semilinear (i.e., $p = 2$) work of Ambrosetti-Brezis-Cerami [1], which is the first to consider problems with concave and convex terms.

The above work was extended to nonlinear problems driven by the p-Laplacian, by Garcia Azorero-Manfredi-Peral Alonso [3] and by Guo-Zhang [4], for $p \geq 2$. In the latter case, the authors also consider reactions of the form

$$\lambda|x|^{q-2}x + g(x),$$

where $g \in C^1(\mathbb{R})$, $g'(x) \geq 0$, $xg(x) \geq 0$, for $x \in \mathbb{R}$ and

$$\lim_{|x| \to 0} \frac{g(x)}{|x|^{p-2}x} = 0, \qquad \lim_{|x| \to \infty} \frac{g(x)}{|x|^{p-2}x} > \lambda_1 .$$

For Dirichlet problems driven by the p-Laplacian and with reactions of more general form we also refer to the following works:

- Boccardo-Escobedo-Peral [2]. The reaction is

  $$\lambda g(x) + x^{r-1}, \quad x \geq 0,$$

  where
  $g : \mathbb{R}_+ \to \mathbb{R}$ continuous, $g(x) \leq \hat{c}x^{q-1}$ for $x \geq 0$ with $\hat{c} > 0$, $1 < q < p < r < p^*$
  and the function $x \to \lambda g(x) + x^{r-1}$ is nondecreasing on $\mathbb{R}_+$.
  They prove the existence of only one positive solution for $\lambda > 0$ suitably small.

- Hu-Papageorgiou[5], where the "convex" ($(p-1)$-superlinear) term is a more general Caratheodory function $f(z,x)$ satisfying the well-known **Ambrosetti-Rabinowitz (AR) condition**:
  " $\exists\, \mu > p$, $M > 0$ such that $\forall x > M$,

  $$0 < \mu F(z,x) \leq f(z,x)x \quad \text{uniformly for a.a. } z \in \Omega."$$

To the best of our knowledge, no bifurcation-type results exist for the **Neumann** problem.

We mention only the work of Wu-Chen[6], where the reaction is of the form $\lambda f(z,x)$, $\lambda > 0$, $f(\cdot,\cdot)$ $(p-1)$-sublinear near infinity in $x \in \mathbb{R}$.

The authors also impose the extra restrictive conditions that $\operatorname*{essinf}_{\Omega} \beta > 0$ and that $N < p$.

They produce three solutions for all $\lambda > 0$ in an open interval. The obtained solutions are not positive.

2 The hypotheses on the perturbation.

(H) : The Carathéodory function $f(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ has $(r-1)$ -polynomial growth with respect to x $(p < r < p*)$. Moreover,

(i) $\lim_{x \to 0+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$

(ii) there exists $\delta_0 > 0$ such that

$$f(z,x) \geq 0 \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in [0,\ \delta_0]$$

and

$\forall \theta > 0$, $\exists\, \hat{\xi}_\theta > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\theta x^{p-1} \quad \text{is increasing on } [0,\theta].$$

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega$$

and

$$\eta_0 \leq \liminf_{x \to +\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \quad \text{uniformly for a.a. } z \in \Omega,$$

where

$$\tau \in \left((r-p) \max\left\{1, \frac{N}{p}\right\},\ p^* \right), \quad q < \tau, \quad \eta_0 > 0$$

Remark 1: Since we are interested in positive solutions and hypotheses H (i), (ii), (iii) involve only the positive semiaxis we may assume that $f(z,x) = 0$ for a.a. $z \in Z$, all $x \leq 0$.

Remark 2: In order to express the " $(p-1)$ -superlinearity" of $f(z,x)$ with respect to x near $+\infty$, instead of the usual in such cases AR-condition, we employ the much weaker conditions H(iii).

Example:

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^{p-1}\left(\ln(x^p+1) + \dfrac{x^p}{x^p+1} \right), & \text{if } x > 0. \end{cases}$$

Note that f satisfies H(iii) but it does not satisfy the AR-condition.

3 Some function spaces

In the study of our problem we will use the following two function spaces

$$C_n^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$$

and

$$W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{||\cdot||},$$

where $||\cdot||$ denotes the Sobolev norm of $W^{1,p}(\Omega)$.
Note that $C_n^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

4 The Euler functional

Let $\varphi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ be the Euler functional for problem (1) defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \frac{\lambda}{q}||u^+||_q^q - \int_\Omega F(z,u)dz,$$

where $F(z,x) = \displaystyle\int_0^x f(z,s)ds$.

Proposition 1 *Under hypotheses (H), $\varphi_\lambda \in C^1\left(W_n^{1,p}(\Omega)\right)$ and each nontrivial critical point of φ_λ is a positive smooth solution of (1).*

The proof is mainly based on the nonlinear regularity theory and also on the nonlinear maximum principle of Vazquez combined with hypothesis H(ii):
" $\forall\theta > 0,\ \exists\, \hat{\xi}_\theta > 0$ such that for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\theta x^{p-1} \quad \text{is increasing on } [0,\theta]."$$

Proposition 2 *Under hypotheses (H), φ_λ satisfies the* **Cerami** *condition (C -condition):*
" Every sequence $\{x_n\}_{n\geq 1} \subseteq X = W_n^{1,p}(\Omega)$ such that

$$\sup_n |\varphi_\lambda(x_n)| < \infty, \quad (1 + ||x_n||)\varphi'_\lambda(x_n) \to 0 \ \textit{in}\ X^* \ \textit{as}\ n \to \infty,$$

has a strongly convergent subsequence "

The proof crucially uses hypothesis H(iii).

5 The bifurcation -type result

$$\begin{cases} -\Delta_p u(z) + \beta(z)|u(z)|^{p-2}u(z) = \lambda|u(z)|^{q-2}u(z) + f(z,u(z)) \\ \text{a.e. in } \Omega, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1) \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \quad (1 < q < p < \infty). \end{cases}$$

Theorem 3 *If hypotheses (H) hold and $\beta \in L^\infty_+(\Omega) \setminus \{0\}$, then there exists $\lambda^* > 0$ such that*

(a) *for $\lambda \in (0,\ \lambda^*)$ problem (1) has at least two positive smooth solutions*

(b) *for $\lambda = \lambda^*$ problem (1) has at least one positive smooth solution*

(c) *for $\lambda > \lambda^*$ problem (1) has no positive solution*

The proof of Theorem 1 may be divided into two parts:
Part I: We consider the set

$$S = \{\lambda > 0 : \text{ problem (1) has a positive smooth } \lambda \text{ -solution}\}$$

and we prove that S is nonempty and bounded from above.
Part II: We prove that $\lambda^* = \sup S$ has the desired properties.

Sketch of the proof of Part I:

Proposition 4 *Under the hypotheses of Th. 3, there exists $\hat{\lambda} > 0$ such that for every $\lambda \in (0,\ \hat{\lambda})$ we can find $\rho_\lambda > 0$ for which we have*

$$\inf [\ \varphi_\lambda(u) : ||u|| = \rho_\lambda\ ] = \eta_\lambda > 0.$$

In order to prove Prop. 4, one shall need hypothesis H(i):

$$\text{“} \lim_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega\text{”}$$

in conjunction with the $(r-1)$ -polynomial growth of $f(z,x)$ with respect to x and also with the inequalities $1 < q < p < r < p^*$.

Proposition 5 *Under the hypotheses of Th. 3, we have*

$$\varphi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty,$$

for each $u \in C_+ \setminus \{0\}$ with $||u||_p = 1$.

The proof of Prop. 5 is based on the p-superlinearity of $F(z,x)$ with respect to x near $+\infty$ (H(iii)) and also on the fact that $q < p$.
Now Prop. 1, 2, 4, 5 via Mountain Pass Theorem yield

Proposition 6 *Under the hypotheses of Th. 3, we have $(0, \hat{\lambda}) \subseteq S$, where $\hat{\lambda}$ is as postulated in Prop. 4. Hence, $S \neq \emptyset$.*

Proposition 7 *Under the hypotheses of Th. 3, the set S is bounded from above.*

For the proof, we shall need the following

Lemma 8 *Let $\beta \in L^\infty(\Omega)_+ \setminus \{0\}$, $u, \tilde{u} \in int\ C_+$ and $R > 0$ such that for a.a. $z \in \Omega$,*

$$-\Delta_p u(z) + \beta(z)u(z)^{p-1} + R \leq -\Delta_p \tilde{u}(z) + \beta(z)\tilde{u}(z)^{p-1}. \qquad (2)$$

Then $u < \tilde{u}$ on $\overline{\Omega}$.

The proof of the above lemma is mainly based on the monotonicity properties of the operator $T : X \to X^*$ $(X = W_n^{1,p}(\Omega))$ induced by the differential operator $u \to -\Delta_p u + \beta(\cdot)|u|^{p-2}u$.

Proof of Prop. 7: The $(p-1)$ -superlinearity of $f(z,x)$ with respect to x near $+\infty$ combined with hypothesis H(ii) enables us to choose $\overline{\lambda} > 0$ large such that

$$\overline{\lambda}x^{q-1} + f(z,x) \geq ||\beta||_\infty x^{p-1} \quad \text{for a.a. } z \in \Omega, \ \text{ all } x \geq 0.$$

Claim: $\overline{\lambda}$ is an upper bound of S.

Indeed, suppose that for some $\lambda > \overline{\lambda}$ our problem has a λ -solution $u \in \text{int } C_+$. Let $m = \min_{\overline{\Omega}} u > 0$. Then for a.a. $z \in \Omega$,

$$\begin{aligned} -\Delta_p u(z) \ &+ \ \beta(z)u(z)^{p-1} \geq ||\beta||_\infty u(z)^{p-1} + (\lambda - \overline{\lambda})u(z)^{q-1} \\ &\geq \ -\Delta_p m + \beta(z)m^{p-1} + (\lambda - \overline{\lambda})m^{q-1} \end{aligned}$$

which implies (see Lemma 8) that $u > m$ on $\overline{\Omega}$ (false!). □

<u>**Sketch of the proof of Part II:**</u>
<u>We begin with two Lemmas:</u>

Lemma 9 *Let $u, \tilde{u} \in int\ C_+$ and $0 < \lambda < \tilde{\lambda}$ such that u is a λ -solution and $\tilde{u}$ is a $\tilde{\lambda}$-solution. If $u \leq \tilde{u}$, then $u < \tilde{u}$ on $\overline{\Omega}$.*

For the proof, we set $\theta = ||\tilde{u}||_\infty$ and we choose $\xi_\theta > 0$ such that $x \to f(z,x) + \xi_\theta x^{p-1}$ is increasing on $[0, \theta]$ (hypothesis H(ii)).

Then (2) holds for

$$\text{"}\beta(\cdot)\text{"} = \beta(\cdot) + \xi_\theta, \quad \text{"}R\text{"} = (\tilde{\lambda} - \lambda)m^{q-1}, \quad m = \min_{\overline{\Omega}} \tilde{u}$$

and now Lemma 8 applies.

Lemma 10 *Let $0 < \lambda < \tilde{\lambda}$ and $\tilde{u} \in int\ C_+$ be a $\tilde{\lambda}$ -solution. Then there exists a λ-solution $u_0 \in int\ C_+$ such that*

$$0 < u_0 < \tilde{u} \ \ on \ \overline{\Omega}, \quad \varphi_\lambda(u_0) < 0.$$

Proof: We consider the following truncation of the reaction:

$$g_\lambda(z,x) = \begin{cases} 0, & \text{if} \quad x \le 0 \\ \lambda x^{q-1} + f(z,x), & \text{if} \quad 0 < x < \tilde{u}(z) \\ \lambda\, \tilde{u}(z)^{q-1} + f(z, \tilde{u}(z)), & \text{if} \quad \tilde{u}(z) \le x. \end{cases}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the C^1-functional $\psi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega G_\lambda(z,u)dz.$$

By using suitable test functions we may show that each critical point of ψ_λ lies in the interval $[0, \tilde{u}]$ and it is also a critical point of the Euler functional φ_λ.

Note that ψ_λ is **coercive** and weakly lower semicontinuous, so we can find $u_0 \in W_n^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_0) = \inf[\ \psi_\lambda(u) : u \in W_n^{1,p}(\Omega)\ ].$$

Then $\psi'_\lambda(u_0) = 0 \Rightarrow u_0 \in [0, \tilde{u}]$ and $\varphi'_\lambda(u_0) = 0$.
Moreover, we may show that for sufficiently small $t > 0$, we have $\psi_\lambda(t) < 0$, so

$$\psi_\lambda(u_0) < 0 = \psi_\lambda(0) \Rightarrow u_0 \ne 0.$$

It follows that u_0 is a positive smooth λ-solution with $\varphi_\lambda(u_0) = \psi_\lambda(u_0) < 0$.
Finally, since $\lambda < \tilde{\lambda}$, we have $u_0 < \tilde{u}$ (see Lemma 5).
Thus, $u_0 \in (0, \tilde{u})$. □

To proceed, set $\lambda^* = \sup S$.

Proposition 11 *If hypotheses of Th. 3 hold and* $\lambda \in (0, \lambda^*)$, *then problem (1) has least two smooth positive solutions*

$$u_0\ ,\ \hat{u} \in intC_+\ , \quad u_0 \ne \hat{u}, \quad u_0 \le \hat{u}, \quad \varphi_\lambda(u_0) < 0.$$

Sketch of the proof:
Let $\lambda \in (0, \lambda^*)$. Choose $\tilde{\lambda} \in (\lambda,\ \lambda^*) \cap S$ and a $\tilde{\lambda}$ -solution $\tilde{u} \in$ int C_+ .
By view of Lemma 10, we may find a λ -solution $u_0 \in$ int C_+ such that

$$0 < u_0 < \tilde{u}, \quad \varphi_\lambda(u_0) < 0.$$

Next, consider the following truncation of the reaction:

$$\hat{f}_\lambda(z,x) = \begin{cases} \lambda u_0(z)^{q-1} + f(z, u_0(z)), & \text{if } x \le u_0(z) \\ \lambda x^{q-1} + f(z,x), & \text{if } u_0(z) < x. \end{cases}$$

Let $\hat{F}_\lambda(z,x) = \displaystyle\int_0^x \hat{f}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega \hat{F}_\lambda(z,u)dz.$$

By using suitable test functions we may show that for each critical point w of $\hat{\varphi}_\lambda$, we have $u_0 \leq w$ and that w is also a critical point of the Euler functional φ_λ.

Evidently, $\hat{\varphi}_\lambda \mid_{[0,\, \widetilde{u}]}$ is coercive and weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in [0,\, \widetilde{u}]$ such that

$$\hat{\varphi}_\lambda(\widetilde{u}_0) = \inf[\, \hat{\varphi}_\lambda(u) : u \in [0,\, \widetilde{u}]\,]\,.$$

Then

$$-\hat{\varphi}'_\lambda(\widetilde{u}_0) \in N_{[\,0,\, \widetilde{u}\,]}(\widetilde{u}_0)$$

where $N_{[\,0,\, \widetilde{u}\,]}(\widetilde{u}_0)$ denotes the normal cone to $[\,0,\, \widetilde{u}\,]$ at $\widetilde{u}_0$.

By using the definition of the normal cone of a closed and convex set combined with our hypotheses, we may show that $\hat{\varphi}'_\lambda(\widetilde{u}_0) = 0$.

It follows that $u_0 \leq \widetilde{u}_0$ and that $\widetilde{u}_0$ is a nontrivial critical point of the Euler functional φ_λ. Hence, $\widetilde{u}_0$ is also a positive smooth λ-solution to our problem.

- If $\widetilde{u}_0 \neq u_0$, we are done.
- Suppose that $\widetilde{u}_0 = u_0$. Since $u_0 \in (0, \widetilde{u})$, we infer that

$$u_0 \quad \text{is a local} \;\; C^1_n(\overline{\Omega}) \;\; - \text{minimizer of } \hat{\varphi}_\lambda\,.$$

It follows from a fact due to Barletta -Papageorgiou (which extends previous results of Brezis - Nirenberg and of Azorero-Manfredi-Alonso) that

$$u_0 \quad \text{is a local} \;\; W^{1,p}_n(\Omega) \;\; - \text{minimizer of } \hat{\varphi}_\lambda\,.$$

Without loss of generality, we may assume that u_0 is an isolated critical point and local minimizer of the functional $\hat{\varphi}_\lambda$.

Then we prove that:

- for some $\rho > 0$,

$$\hat{\varphi}_\lambda(u_0) < \inf[\, \hat{\varphi}_\lambda(u) : ||u - u_0|| = \rho\,]$$

- for every $u \in$ int C_+ with $||u||_p = 1$,

$$\hat{\varphi}_\lambda(tu) \to -\infty, \quad \text{as } t \to +\infty$$

- $\hat{\varphi}_\lambda$ satisfies the C -condition

Arguing via Mountain Pass Theorem we may find a critical point $\hat{u}$ of $\hat{\varphi}_\lambda$ such that $\hat{u} \neq u_0$.
It follows that $u_0 \leq \hat{u}$ and that $\hat{u}$ is a nontrivial critical point of the Euler functional φ_λ.
Hence, $\hat{u}$ is a second positive smooth λ-solution to our problem. □

Proposition 12 *If hypotheses of Th. 3 hold, then for $\lambda = \lambda^*$, problem (1) has at least one smooth positive solution.*

The key ingredient in the proof of Proposition 12, is the following

Lemma 13 *Let $S' \subseteq S$ be nonempty and bounded from below with* $\inf S' > 0$ *and $B \subseteq int\ C_+$ be $||\cdot||_\infty$ -bounded. Then there exists $w \in\ int\ C_+$ such that for each $\lambda \in S'$ and for each λ-solution $u \in B$, we have $w \leq u$.*

Sketch of the proof of Prop. 12: Choose a nondecreasing sequence $(\lambda_n) \subseteq S$ such that $\lambda_n \uparrow \lambda^*$. By view of Prop.11, we may find $\{u_n\}_{n\geq 1} \subseteq$ int C_+ such that

$$\varphi'_{\lambda_n}(u_n) = 0, \quad \varphi_{\lambda_n}(u_n) < 0, \quad \text{for all } n \geq 1.$$

Arguing in a similar way as in the proof of the Cerami condition, we may show (by passing to subsequences) that

$$u_n \to u_*, \text{ strongly in } W_n^{1,p}(\Omega).$$

Then nonlinear regularity theory guarantees that

$$\sup_n ||u_n||_\infty < \infty$$

and that u_* is a smooth λ^*-solution.

Now Lemma 13 asserts that for some $w \in$ int C_+, we have $w \leq u_n$, $n \geq 1$. Thus, $w \leq u_*$, so $u_* \in$ int C_+.

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